Section 2-7, Mathematics 108

Combining Functions

Assuming we have two functions

$f(x)$

and

$g(x)$

that both return numerical values on the same domain.

We can combine these functions in various way to create new functions.

For example we could create a new function

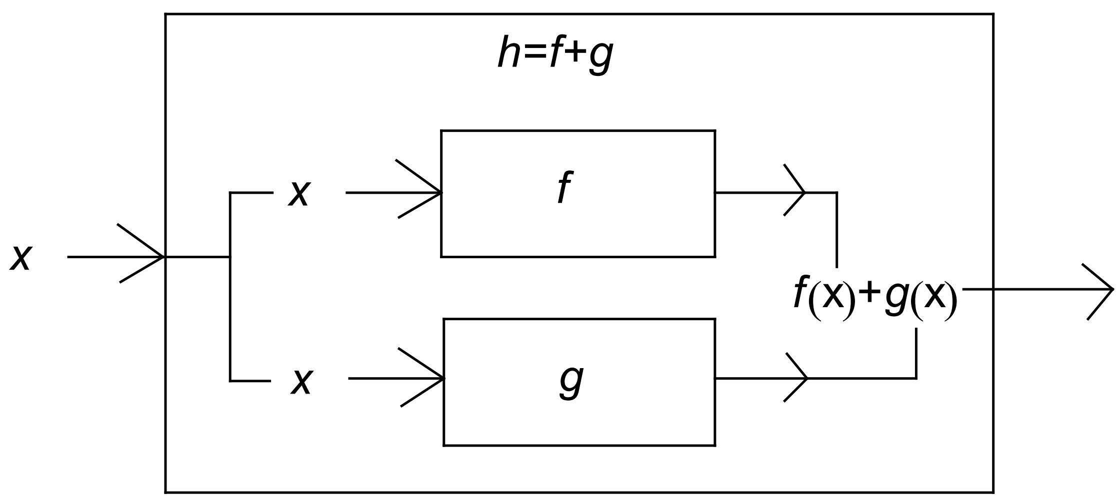
$h(x) = f(x) + g(x)$

What this new function $h(x)$ does is for each x in the domain, returns a value that is the sum of what the two other functions $f(x)$ and $g(x)$ returns.

Which adds the results of f and g.

Here is a way to look at it as a machine.

The h machine just has the f and g machines hidden inside.



If the function rule describing *f* and *g* is algebraic, then it is easy to find the rule for *h*.

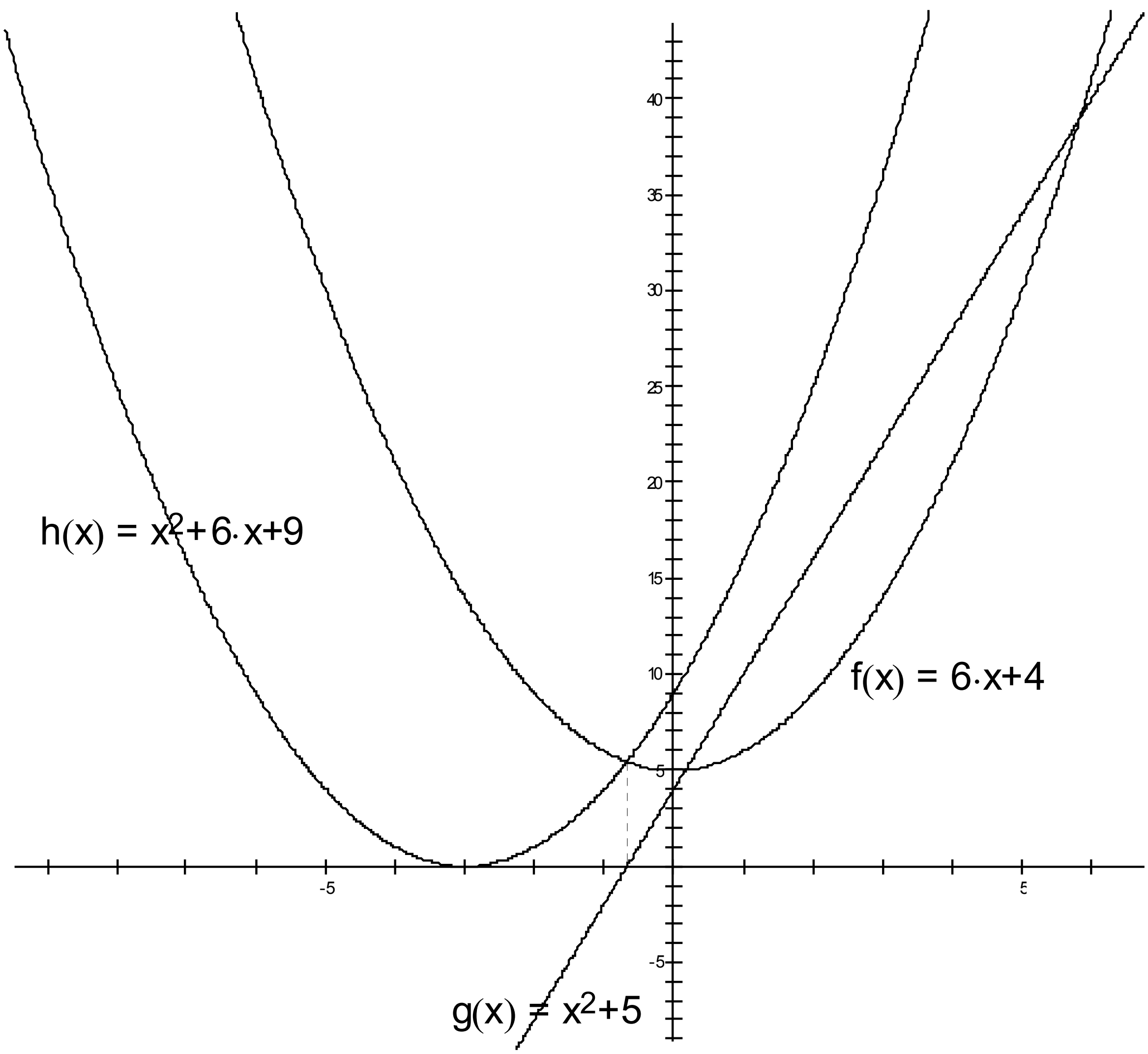
Example:

$$f(x) = 6x + 5$$
$$g(x) = x^2 + 4$$

then the rule for *h* is

$$h(x) = f(x) + g(x) = 6x + 5 + x^2 + 4 = x^2 + 6x + 9$$

You might want to see what this looks like graphically:

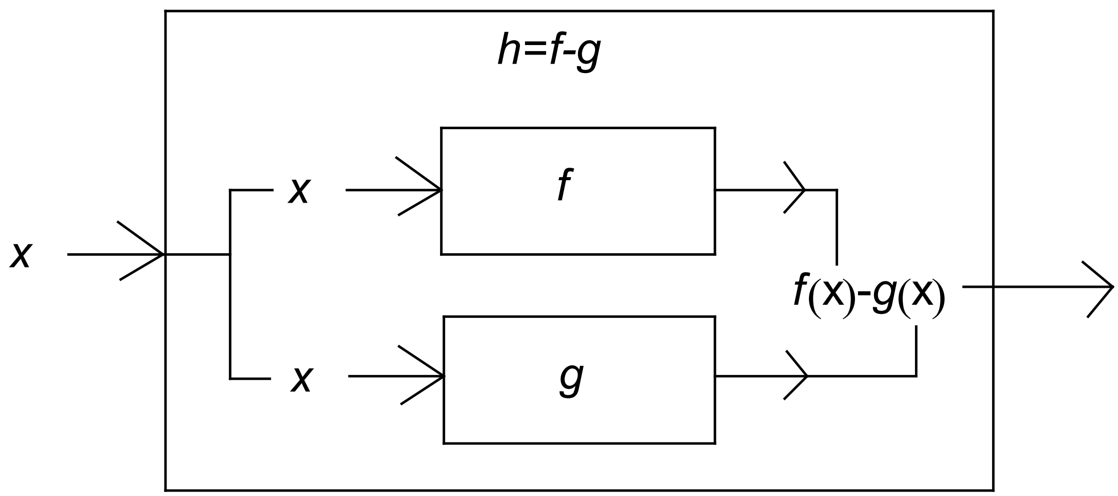


Note the *x* at $x = -\frac{2}{3}$ $f(x) = 0$ (dotted line) and $g(x)$ and $h(x)$ intersect.

We can combine these functions in other ways.

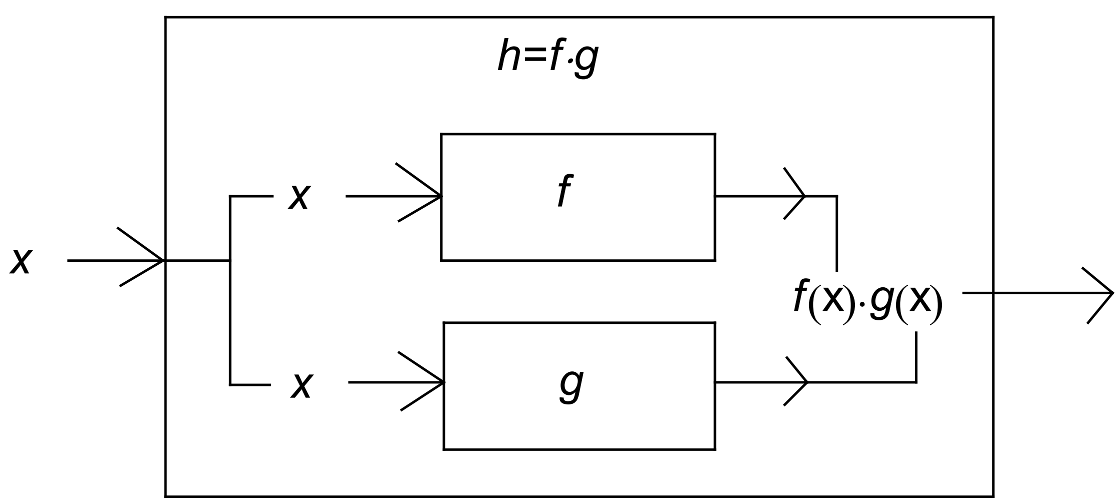
By Subtracting

$h(x) = f(x) - g(x) = -x^2 + 6x + 1$



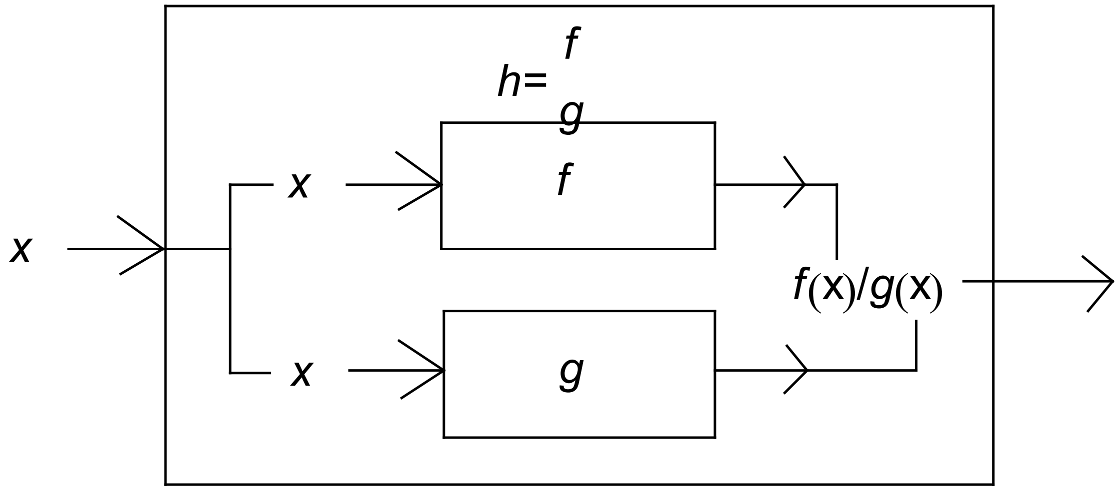
By Multiplying

$h(x) = f(x)g(x) = (6x+5)(x^2+4) = 6x^3 + 5x^2 + 24x + 20$



And by Dividing

$$h(x)=\frac{f(x)}{g(x)}=\frac{6x+5}{x^2+4}$$



Keep in mind that what we've been doing is adding/subtracting/multiplying and dividing the function rule, but we have to be careful about the domain of the function *h.*

In all four cases our domain can be at most the domain of the intersection of the two original domains.

In the case of dividing if $g(x)$ is zero for some *x's* then we will have to also exclude these values from *h*'s domain.

Here's a summary of how the domain of the new function might change.

$$(f+g)(x)=f(x)+g(x) \qquad \textit{Domain } A\cap B$$
$$(f-g)(x)=f(x)-g(x) \qquad \textit{Domain } A\cap B$$
$$(fg)(x)=f(x)g(x) \qquad \textit{Domain } A\cap B$$
$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)} \qquad \textit{Domain } \{x\in A\cap B \mid g(x)\neq 0\}$$

Examples: Let $f(x)=\dfrac{1}{x-2}$ *and* $g(x)=\sqrt{x}$, find the value of the new function at 4.

$$(f+g)(x)=\frac{1}{x-2}+\sqrt{x} \quad Domain\{x \mid x \geq 0 \text{ and } x \neq 2\}$$
$$(f+g)(4)=\frac{1}{4-2}+\sqrt{4}=\frac{1}{2}+2=\frac{5}{2}$$

$$(f-g)(x)=\frac{1}{x-2}-\sqrt{x} \quad Domain\{x \mid x \geq 0 \text{ and } x \neq 2\}$$
$$(f+g)(4)=\frac{1}{4-2}-\sqrt{4}=\frac{1}{2}-2=-\frac{3}{2}$$

$$(fg)(x)=\frac{\sqrt{x}}{x-2} \quad Domain\{x \mid x \geq 0 \text{ and } x \neq 2\}$$
$$(f+g)(4)=\frac{2}{2}=1$$

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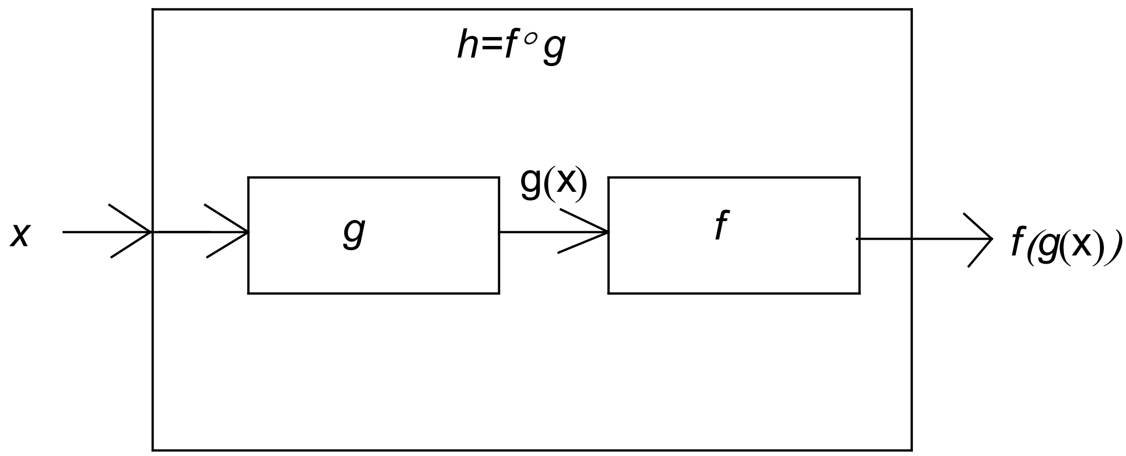
$$\left(\frac{f}{g}\right)(x)=\frac{\sqrt{x}}{\frac{1}{x-2}}=\sqrt{x}(x-2) \quad Domain\{x \mid x \geq 0 \text{ and } x \neq 2\}$$
$$\left(\frac{f}{g}\right)(4)=\sqrt{4}(4-2)=4$$

Note, $\left(\dfrac{f}{g}\right)(2)=\sqrt{2}(2-2)=0$ why does the domain still exclude 2?

Composition of functions

One more extremely important way to combine functions is using composition.

When using composition of functions, the output of one function becomes the input to another.



$g(x)=x^2 \quad f(x)=x+1$

$$\left|\begin{pmatrix}1\\2\\3\end{pmatrix} \rightarrow \begin{pmatrix}1\\4\\9\end{pmatrix} \rightarrow \begin{pmatrix}2\\5\\10\end{pmatrix}\right.$$

We write this either as

$f(g(x))$

or

$(f \circ g)(x)$

Note that you must be very careful about the order in which you combine.

$$f(g(x)) = f(x^2) = x^2 + 1$$
$$g(f(x)) = g(x+1) = (x+1)^2 = x^2 + 2x + 1$$

And these are very different! So in general $(g \circ f)(x) \neq (f \circ g)(x)$

You must also be careful about the domains and ranges.

In the first example, the range of *g* might have values excluded from the domain of *f*.

Extreme Example:

What is the domain of $g\left(f\left(x\right)\right)$ if

$f\left(x\right)=-x^2 and\ g\left(x\right)=\sqrt{x}$

Domain = {0}

Example:

$f\left(x\right)=x^3 and\ g\left(x\right)=\sqrt{x-3}$

What are the domains of $f\left(g\left(x\right)\right)$ and $g\left(f\left(x\right)\right)$ and what are $f\left(g\left(4\right)\right)$ and $g\left(f\left(4\right)\right)$

$f\left(g\left(x\right)\right)=f\left(\sqrt{x-3}\right)=\left(\sqrt{x-3}\right)^3=\left(x-3\right)^{3/2}$

Domain $x\geq 3$

$f\left(4\right)=\left(4-3\right)^{3/2}=1^{3/2}=1$

$g\left(f\left(x\right)\right)=g\left(x^3\right)=\sqrt{x^3-3}$

Domain $x\geq\sqrt[3]{3}$

$g\left(f\left(4\right)\right)=\sqrt{4^3-3}=\sqrt{61}$